92

Birzeit University- Mathematics Department
Calculus II-Math 132

Second Exam | Spring 2012/2013
Name(Arabic):... | Number:...1100234
Instructor of Discussion(Arabic):.................... | Section:....8:00.....

Time: 80 Minutes | **There are 4 questions in 7 pages.**

Question 1. (54%) Circle the correct answer:

1. The sequence $a_n = \sqrt{n+1} - \sqrt{n}$
   (a) Diverges.
   (b) Converges to 0.
   (c) Converges to 1.
   (d) None.

2. The series $\sum_{n=2}^{\infty} \frac{\ln n}{n^3}$
   (a) Diverges by nth term test.
   (b) Converges by limit comparison with $\sum_{n=2}^{\infty} \frac{1}{n^3}$.
   (c) Converges by direct comparison with $\sum_{n=2}^{\infty} \frac{1}{n^3}$.
   (d) Converges by direct comparison with $\sum_{n=2}^{\infty} \frac{1}{n^2}$.

3. The series $\sum_{n=1}^{\infty} \frac{n}{n+1}$
   (a) Converges to 1.
   (b) Converges by ratio test.
   (c) Diverges by ratio test.
   (d) Diverges by nth term test.

4. $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{2}{3}\right)^n =$
   (a) $\frac{3}{5}$.
   (b) $\frac{2}{3}$.
   (c) $\frac{2}{5}$.
   (d) Diverges by alternating series test.

5. The series $\sum_{n=1}^{\infty}\left(\frac{1}{e^{n}+e^{-n}}\right)$

(a) Converges to 1.

(b) Is a geometric series.

(c) Converges by integral test.

(d) Diverges by nth term test.

6. The series $\sum_{n=0}^{\infty}\frac{3^{n}}{2^{n}+5^{n}} < \frac{3^n}{5^n}$

(a) Diverges by direct comparison with $\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^{n}$.

(b) Diverges by nth term test.

(c) Converges by direct comparison with $\sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^{n}$.

(d) Converges by direct comparison with $\sum_{n=0}^{\infty}\left(\frac{3}{2}\right)^{n}$.

7. $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^{p}}$

(a) Converges absolutely if $p \geq 1$.

(b) Converges conditionally if $0 < p \leq 1$.

(c) Converges absolutely if $0 < p \leq 1$.

(d) Diverges.

8. The series $\sum_{n=1}^{\infty}\frac{n!}{(2n)!}$

(a) Converges by ratio test.

(b) Diverges by ratio test.

(c) Diverges by integral **test**.

(d) Diverges by nth term test.

9. The series $\sum_{n=1}^{\infty}\frac{n^{2}}{n^{n}}$

(a) Diverges by nth term test.

(b) Diverges by nth root test.

(c) Converges by nth root test.

(d) Converges by alternating series test.

10. The Maclaurin series generated by the function e^{x^2} is

(a) $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$.

(b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

(c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$.

(d) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n}$.

11. One of the following improper integrals converges

(a) $\int_1^{\infty} \frac{e^x}{x} dx$.

(b) $\int_0^1 \frac{dx}{x}$. div

(c) $\int_0^1 \frac{dx}{\sqrt{x}}$.

(d) $\int_2^{\infty} \frac{dx}{\ln x}$. div

12. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, $a_n, b_n > 0$ for all n then

(a) $\sum a_n$ and $\sum b_n$ both converge or both diverge.

(b) If $\sum a_n$ converges then $\sum b_n$ converges.

(c) If $\sum b_n$ converges then $\sum a_n$ converges.

(d) If $\sum b_n$ diverges then $\sum a_n$ diverges.

$a_n < b_n$ conv conv

13. The sequence $a_n = (1 + \frac{1}{n})^{-n}$

(a) converges to 1.

(b) converges to e.

(c) converges to $-e$.

(d) converges to e^{-1}.

$\frac{1}{(1+\frac{1}{n})^n}$ $\frac{1}{e}$

14. The radius of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ is

(a) $R = 1$.

(b) $R = \infty$.

(c) $R = 0$.

(d) $R = e$.

$\frac{|x|}{n} = 0$

conv on $(-\infty, \infty)$

$R = \infty$

$\frac{1}{1-x} = 3$

$3 - 3x = 1$

15. $\sum_{n=1}^{\infty} x^n = 2$ if $x =$

$x + x^2 + x^3 = \frac{x}{1-x} = 2$

$2 = \frac{1}{1-x}$

$2 - 2x = 1$

$2 - 2x = x$

(a) $\frac{1}{2}$.
(b) $\frac{2}{3}$.
(c) $\frac{1}{3}$.
(d) $\frac{1}{4}$.

16. The series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n+1}}$

$\frac{x}{1-x} = 1$

$1 - x = x$

$2x = 1 \rightarrow \boxed{x = \frac{1}{2}}$

(a) Diverges by direct comparison with $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$.

(b) Diverges by nth term test.

(c) Converges by limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$.

(d) Diverges by limit comparison test with $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$.

$\frac{\frac{1}{\sqrt{n+1}}}{\frac{1}{\sqrt{n}}}$

$\frac{\sqrt{n}}{\sqrt{n+1}} = 1$

17. If we approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ using the 8th partial sum s_8 then the error in this approximation

error $< S_9$

(a) is less than $\frac{1}{64}$.
(b) is less than $\frac{1}{81}$.
(c) is greater than $\frac{1}{64}$.
(d) is greater than $\frac{1}{81}$.

$e < S_9$

$|e| < \frac{(-1)^{10}}{81} < \frac{1}{81}$

18. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$

$\left|\frac{1}{n^3+1}\right| < \frac{1}{n^3}$

conv.

(a) Diverges.
(b) Converges absolutely.
(c) Converges conditionally.
(d) Diverges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$.

48

Question 2(16%) Find the radius and interval of convergence of the following series

$$\sum_{n=0}^{\infty} \frac{nx^n}{4^n(n^2+1)}$$

Then, specify the points at which the series converges conditionally, absolutely.

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)\,|x|^{n+1}}{4^{n+1}\,((n+1)^2+1)} \cdot \frac{4^n\,(n^2+1)}{n\,|x|^n}$

$= \lim_{n\to\infty} \frac{|x|\,(n+1)(n^2+1)}{4\,((n+1)^2+1)\,n} = \frac{|x|}{4}$

$\frac{|x|}{4} < 1$

$|x| < 4$

$-4 < x < 4$

$\boxed{x=-4} \to \sum_0^{\infty} \frac{n(-4)^n}{4^n(n^2+1)} = \sum_0^{\infty} \frac{n(-1)^n(4)^n}{4^n(n^2+1)} = \sum_0^{\infty} \frac{(-1)^n\, n}{n^2+1}$

$|u_n| = \sum \frac{n}{n^2+1}$

$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = 1$ div by nth test

$\sum_0^{\infty} \frac{(-1)^n\, n}{n^2+1}$ conv by A.S.Th

① $U_{n+1} < U_n$
② U_n positive
③ $\lim_{n\to\infty} U_n \to 0$

$\boxed{x=4} \to \sum_0^{\infty} \frac{n\,4^n}{4^n(n^2+1)} = \sum_0^{\infty} \frac{n}{n^2+1}$

$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$ div by L.C.T by compare with $\frac{1}{n}$ which is div by p-test

الله معك

Question 3(15%) Answer the questions below

(a) Find the sum of the series $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$.

$$= \sum_{2}^{\infty} \frac{2}{(n+1)(n-1)} = \sum_{2}^{\infty} \frac{1}{(n+1)} - \frac{1}{(n-1)}$$

$$= \left(\frac{1}{3} - \frac{1}{1}\right) + \left(\frac{1}{4} - \frac{1}{2}\right) + \left(\frac{1}{5} - \frac{1}{3}\right) + \left(\frac{1}{6} - \frac{1}{4}\right) + \left(\frac{1}{7} - \frac{1}{5}\right) + \frac{1}{(n+1)} - \frac{1}{(n-1)}$$

$$= \lim_{k\to\infty} -1 - \frac{1}{2} + \frac{1}{n+1} = \lim_{k\to\infty} -\frac{3}{2} + \frac{1}{n+1} = -\frac{3}{2}$$

4

(b) Determine whether the integral $\int_{1}^{\infty} \frac{dx}{e^x+x^2}$ converges or diverges.

$$\int_1^{\infty} \frac{dx}{e^x+x^2} < \frac{1}{x^2}$$

$\int_1^{\infty} \frac{1}{x^2}$ is conv by p-test

so $\int_1^{\infty} \frac{dx}{e^x+x^2}$ is conv by D.C.T with $\int_1^{\infty} \frac{1}{x^2}$

5

(c) Determine whether the integral $\int_{0}^{1} \frac{2dx}{x(x+2)}$ converges or diverges.

$$\frac{2}{x^2+2x} < \int_0^1 \frac{2}{x^2} \quad \text{div.}$$

$$\int_0^1 \frac{2dx}{x^2+2x} = \lim_{a\to 0^+} \int_a^1 \frac{2\,dx}{x^2+2x} = \lim_{a\to 0^+} \int_a^1 \frac{2\,dx}{(x)(x+2)}$$

$$\Rightarrow \int_a^1 \frac{2\,dx}{x(x+2)} \Rightarrow \frac{2}{x(x+2)} = \frac{A}{x} + \frac{B}{(x+2)}$$

$$\frac{1}{x^2} \qquad \frac{x^2}{x^2+2x} \to 1 \quad \text{both dive}$$

$$2 = A(x+2) + Bx$$

$$x=0 \to \boxed{A=1}, \quad x=-2 \to \boxed{B=-1}$$

$$\Rightarrow \int_a^1 \frac{1}{x} + \frac{-1}{x+2}$$

$$= \ln|x| - \ln|x+2| \Big|_a^1 = \ln 1 - \ln 2 - \ln a + \ln(a+2)$$

خلاف الو ردی ←

4

Question 4(15%) Answer the following:

(a) Find the Taylor series generated by the function $f(x) = 2^x$ at $a = 1$.

$$\sum_{0}^{\infty} \frac{f^{n}(a)(x-1)}{n!} = f(1) + f'(1)(x-1) + f''(1)\frac{(x-1)^2}{2!} + f'''(1)\frac{(x-1)^3}{3!}$$

$$= 2 + 2\ln 2\,(x-1) + \frac{2(\ln 2)^2(x-1)^2}{2!} + \frac{2(\ln 2)^3(x-1)^3}{3!}$$

$$= \sum_{0}^{\infty} \frac{2(\ln 2)^n (x-1)^n}{n!}$$

$f(x) = 2^x \to f(1) = 2$

$f'(x) = 2^x \ln 2 \to f'(1) = 2\ln 2$

$f''(x) = (\ln 2)^2 2^x \to f''(1) = 2(\ln 2)^2$

$f'''(x) = (\ln 2)^3 2^x \to f'''(1) = (\ln 2)^3$ $2x$

(b) Use the fact that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$ to answer the following questions:

(i) Find the Maclaurin series of the function $\frac{1}{1+x^2}$.

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$\sum_{0}^{\infty} (-x^2)^n = \sum_{0}^{\infty} (-1)^n x^{2n}$$

(ii) Use (i) to find the Maclaurin series of the function $\tan^{-1} x$.

$$\int \frac{1}{1+x^2} = \tan^{-1} x$$

$$\int \sum_{0}^{\infty} (-1)^n x^{2n} = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

(iii) What is the interval of convergence of the series in (i) and (ii).

series(i) $\to \sum_{0}^{\infty} (-1)^n x^{2n}$ is div by nth test ~~[illegible]~~

Conv at $\boxed{x=0}$ ~~$\lim_{n\to\infty} a_n$ [illegible]~~

Series(ii) $\to \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$

$\lim_{n\to\infty} \frac{|x|^{2n+3}}{2n+3} \cdot \frac{2n+1}{|x|^{2n+1}} \Rightarrow \lim_{n\to\infty} \frac{|x|^2 (2n+1)}{(2n+3)} \to |x|^2$ / $|x|^2 < 1$ $(|x| < 1)$ $-1 < x < 1$

Conv at $-1 < x < 1$